NONSTEADY TEMPERATURE FIELD IN THERMOSENSITIVE BODY WITH DISCONTINUOUS BOUNDARY CONDITIONS OF THE SECOND KIND

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A method of determining the nonsteady temperature fields in bodies with discontinuous boundary conditions is proposed, taking account of the temperature dependence of the thermophysical characteristics.

It was noted in [1, 2] that nondestructive methods of determining thermophysical characteristics are based on the solution of two-dimensional nonsteady heat-conduction problems with discontinuous boundary conditions of the second kind. The nonsteady temperature fields in thin isotropic plates with heat transfer were determined in [3, 4] for the case of discontinuous boundary conditions of the second kind. In [5], these results were generalized to anisotropic plates. These investigations were reported in [6, 7]. These results, with a heat-transfer coefficient from the side surfaces of the plates $\alpha = 0$ (heat-insulated plates), lead to expressions for the nonsteady two-dimensional temperature fields in a halfspace and a rectangular wedge with heat transfer over a strip region of its boundary surface. Thanks to the use of generalized functions, closed solutions that are convenient for investigation and common to the whole region of definition are obtained. The thermophysical characteristics (TPC) of the given plates are assumed to be independent of the temperature. Such bodies are said to be nonthermosensitive, and bodies with temperature-dependent TPC to be thermosensitive [8].

Now consider a thermosensitive layer, at the boundary surface $z = 0$ of which a discontinuous boundary condition of the second kind is specified

$$\lambda(t)\frac{\partial t}{\partial z} = -q(\tau)N(r) \text{ when } z = 0, \tag{1}$$

where

$$N(r) = S(r - R + h) - S(r - R - h);$$

$$S(\zeta) = \begin{cases} 1, & \zeta > 0, \\ 0{,}5, & \zeta = 0, \\ 0, & \zeta < 0 \end{cases} \quad \text{is a symmetric unit function [9].}$$

The surface $z = l$ of the layer is assumed to be heat-insulated

$$\lambda(t)\frac{\partial t}{\partial z} = 0 \text{ when } z = l. \tag{2}$$

The initial temperature and the temperature at infinity are

$$t|_{\tau=0} = 0, \ t|_{r\to\infty} = 0. \tag{3}$$

The heat-conduction equation for determining the nonsteady temperature field in the given layer takes the form

$$\operatorname{div}[\lambda(t)\operatorname{grad} t] = c_v(t)\dot{t}. \tag{4}$$

For many materials [8, 10-13], the thermal diffusivity a is constant or varies only slightly as a function of the temperature. In this case, the boundary problem in Eqs. (1)-(4) is completely linearized by means of the Kirchhoff variable

$$\vartheta = \frac{1}{\lambda_0}\int_0^t \lambda(\zeta)\,d\zeta \tag{5}$$

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and takes the form

$$\left.\frac{\partial\vartheta}{\partial z}\right|_{z=0} = -\frac{q(\tau)}{\lambda_0} N(r), \quad \left.\frac{\partial\vartheta}{\partial z}\right|_{z=l} = 0, \tag{6}$$

$$\vartheta|_{\tau=0} = 0, \quad \vartheta|_{r\to\infty} = 0, \tag{7}$$

$$\frac{\partial^2\vartheta}{\partial r^2} + \frac{1}{r}\frac{\partial\vartheta}{\partial r} + \frac{\partial^2\vartheta}{\partial z^2} = \frac{\dot\vartheta}{a}. \tag{8}$$

Applying an integral Laplace transformation with respect to the time and Hankel transformation with respect to r to Eqs. (8) and (6), taking account of Eq. (7), and using handbook data [14], it is found that

$$\frac{d^2\bar\vartheta}{dz^2} = \gamma^2\bar\vartheta, \tag{9}$$

$$\left.\frac{d\bar\vartheta}{dz}\right|_{z=0} = -\frac{\tilde q(s)}{\lambda_0}[\Phi(\xi, R+h) - \Phi(\xi, R-h)], \quad \left.\frac{d\bar\vartheta}{dz}\right|_{z=l} = 0, \tag{10}$$

where

$$\gamma = \sqrt{\xi^2 + \frac{s}{a}}; \quad \tilde q(s) = \int_0^\infty q(\tau)\exp(-s\tau)\,d\tau; \quad \dot\vartheta = \frac{\partial\vartheta}{\partial\tau};$$

$$\Phi(\xi, \zeta) = \frac{\zeta}{\xi} J_1(\xi\zeta); \quad \bar\vartheta(\xi, z, s) = \int_0^\infty\int_0^\infty r\vartheta(r, z, \tau) J_0(\xi r)\exp(-s\tau)\,dr\,d\tau.$$

The solution of Eq. (9) with the boundary conditions in Eq. (10) is written in the form

$$\bar\vartheta = \frac{\tilde q(s)}{\lambda_0\gamma}\frac{\operatorname{ch}\gamma(z-l)}{\operatorname{sh}\gamma l}[\Phi(\xi, R+h) - \Phi(\xi, R-h)]. \tag{11}$$

If the ring is narrow, then passing to the limit as $h \to 0$ in Eq. (6), taking into account that $\lim_{h\to 0} N(r)/2h = \delta(r - R)$ [15, 16], and maintaining $Q(\tau) = 2hq(\tau)$ constant, it is found that

$$\left.\frac{\partial\vartheta}{\partial z}\right|_{z=0} = -\frac{Q(\tau)}{\lambda_0}\delta(r-R). \tag{12}$$

After applying Laplace–Hankel transformation to Eq. (12), the result is

$$\left.\frac{d\bar\vartheta}{dz}\right|_{z=0} = -\frac{\tilde Q(s)}{\lambda_0} J_0(\xi R).$$

The solution of Eq. (9) for this case is written in the form

$$\bar\vartheta = \frac{\tilde Q(s) R}{\lambda_0\gamma}\frac{\operatorname{ch}\gamma(z-l)}{\operatorname{sh}\gamma l} J_0(\xi R). \tag{13}$$

Passing to the limit as $l \to \infty$ in Eqs. (11) and (13), the expression is obtained for the transform of the Kirchhoff variable for a halfspace

$$\bar\vartheta = \frac{\tilde q(s)}{\lambda_0\gamma}\exp(-\gamma z)[\Phi(\xi, R+h) - \Phi(\xi, R-h)], \tag{14}$$

$$\bar\vartheta = \frac{\tilde Q(s) R}{\lambda_0\gamma}\exp(-\gamma z) J_0(\xi R). \tag{15}$$

Reverting to the original from the transform in Eqs. (11) and (13)-(15) and taking account of handbook data [17, 18], the following expression is obtained for the Kirchhoff variable in an annular heating region of arbitrary width 2h and in the case of heating over a narrow ring for a layer and a halfspace, respectively

$$\vartheta = \frac{a}{\lambda_0 l}\int_0^\tau q(\tau-\tau_0)\,\vartheta_3\left(\frac{z}{2l}, \frac{a\tau_0}{l^2}\right)[\varphi(\tau_0, R_-) - \varphi(\tau_0, R_+)]\,d\tau_0, \tag{16}$$

$$\vartheta = \frac{R}{2\lambda_0 l}\int_0^\tau Q(\tau-\tau_0)\,\vartheta_3\left(\frac{z}{2l},\quad \frac{a\tau_0}{l^2}\right)\exp(-\Omega-\omega)\,I_0(2\sqrt{\Omega\omega})\,\frac{d\tau}{\tau_0}, \tag{17}$$

$$\vartheta = \sqrt{\frac{a}{\pi}}\int_0^\tau \frac{q(\tau-\tau_0)}{\lambda_0\sqrt{\tau_0}}\exp(-b)\,[\varphi(\tau_0,\quad R_-)-\varphi(\tau_0,\ R_+)]\,d\tau_0, \tag{18}$$

$$\vartheta = \frac{R}{2\lambda_0\sqrt{\pi a}}\int_0^\tau Q(\tau-\tau_0)\exp(-\Omega-\omega-b)\,I_0(2\sqrt{\Omega\omega})\,\frac{d\tau_0}{\tau_0^{3/2}}, \tag{19}$$

where

$$\varphi(\tau_0,\ \zeta) = \sum_{k=0}^{\infty}\frac{(-b_0)^{k+1}}{\zeta(k+1)!}\,{}_2F_1\left(-k,\ -1-k;\ 1;\ \frac{r^2}{\zeta^2}\right);\quad R_\pm = R\pm h;$$

$$\Omega = \frac{R^2}{4a\tau_0};\qquad \omega = \frac{r^2}{4a\tau_0};\qquad b = \frac{z^2}{4a\tau_0};\qquad b_0 = \frac{\zeta^2}{4a\tau_0}.$$

If Eqs. (11) and (14) remain in integral form

$$\bar\vartheta = \frac{\tilde q(s)}{\lambda_0\gamma}\,\frac{\operatorname{ch}\gamma(z-l)}{sh\,\gamma l}\int_{R_-}^{R_+} r_0 J_0(\xi r_0)\,dr_0, \tag{20}$$

$$\bar\vartheta = \frac{\tilde q(s)}{\lambda_0\gamma}\exp(-\gamma z)\int_{R_-}^{R_+} r_0 J_0(\xi r_0)\,dr_0, \tag{21}$$

inversion from the transform to the original in Eqs. (20) and (21) leads to simpler expressions for the Kirchhoff variable

$$\vartheta = \frac{a}{\lambda_0 l}\int_0^\tau q(\tau-\tau_0)\,\vartheta_3\left(\frac{z}{2l},\quad \frac{a\tau_0}{l^2}\right)[J(\Omega_-,\ \omega)-J(\Omega_+,\ \omega)]\,d\tau_0, \tag{22}$$

$$\vartheta = \sqrt{\frac{a}{\pi}}\int_0^\tau \frac{q(\tau-\tau_0)}{\lambda_0\sqrt{\tau_0}}\exp(-b)\,[J(\Omega_-,\ \omega)-J(\Omega_+,\ \omega)]\,d\tau_0, \tag{23}$$

where $J(\zeta,\ \omega) = 1-\int_0^\zeta \exp(-\omega-\omega_0)\,I_0(2\sqrt{\omega_0\omega})\,d\omega_0$ is a fundamental function [10]; $\omega_0 = r_0^2/4a\tau_0$; $\Omega_\pm = R_\pm^2/4a\tau_0$.

At the surface of the layer and the halfspace z = 0, the Kirchhoff variable for the case of an arbitrary width and a narrow ring takes the form, respectively

$$\vartheta|_{z=0} = \frac{a}{\lambda_0 l}\int_0^\tau q(\tau-\tau_0)\,\vartheta_3\left(0,\ \frac{a\tau_0}{l^2}\right)[J(\Omega_-,\ \omega)-J(\Omega_+,\ \omega)]\,d\tau_0, \tag{24}$$

$$\vartheta|_{z=0} = \frac{R}{2\lambda_0 l}\int_0^\tau Q(\tau-\tau_0)\,\vartheta_3\left(0,\ \frac{a\tau_0}{l^2}\right)\exp(-\Omega-\omega)\,I_0(2\sqrt{\Omega\omega})\,\frac{d\tau_0}{\tau_0}, \tag{25}$$

$$\vartheta|_{z=0} = \sqrt{\frac{a}{\pi}}\int_0^\tau \frac{q(\tau-\tau_0)}{\lambda_0\sqrt{\tau_0}}[J(\Omega_-,\ \omega)-J(\Omega_+,\ \omega)]\,d\tau_0, \tag{26}$$

$$\vartheta|_{z=0} = \frac{R}{2\lambda_0\sqrt{\pi a}}\int_0^\tau Q(\tau-\tau_0)\exp(-\Omega-\omega)\,I_0(2\sqrt{\Omega\omega})\,\frac{d\tau_0}{\tau_0^{3/2}}. \tag{27}$$

The solution of the problem for nonthermosensitive regions is obtained from Eqs. (16)-(19), (22), and (23) by replacing ϑ, λ_0 by t, λ. In particular, the following expression is obtained at the surface of a halfspace subjected to the action of a nonsteady heat flux over an annular region of arbitrary width

$$t|_{z=0}=\sqrt{\frac{a}{\pi}}\int_0^\tau \frac{q(\tau-\tau_0)}{\lambda\sqrt{\tau_0}}\left[J(\Omega_-,\ \omega)-J(\Omega_+,\ \omega)\right]d\tau_0. \tag{28}$$

For this case, the solution of the problem at the surface of a halfspace was found in [2] by another method, in a different form.

Passing to the limit as $R_- \to 0$ in Eqs. (24), (26), and (28), a solution corresponding to heating of a halfspace by a heat flux of intensity $q(\tau)$ over a round region of radius R_+ is obtained. In particular, as $R_- \to 0$, Eq. (26) gives

$$\vartheta|_{z=0}=\sqrt{\frac{a}{\pi}}\int_0^\tau \frac{q(\tau-\tau_0)}{\lambda_0\sqrt{\tau_0}}\left[1-J(\Omega_+,\ \omega)\right]d\tau_0. \tag{29}$$

The Kirchhoff variable at the points r = 0, r = R on the surface of the halfspace takes the form

$$\begin{aligned}\vartheta_0=\vartheta\Big|_{\substack{z=0,\\ r=0}}&=\sqrt{\frac{a}{\pi}}\int_0^\tau \frac{q(\tau-\tau_0)}{\lambda_0\sqrt{\tau_0}}\left[J(\Omega_-,\ 0)-J(\Omega_+,\ 0)\right]d\tau_0,\\ \vartheta\Big|_{\substack{z=0\\ r=R}}&=\sqrt{\frac{a}{\pi}}\int_0^\tau \frac{q(\tau-\tau_0)}{\lambda_0\sqrt{\tau_0}}\left[J(\Omega_-,\ \Omega)-J(\Omega_+,\ \Omega)\right]d\tau_0,\end{aligned} \tag{30}$$

where $J(\Omega_\pm,\ 0) = \exp(-\Omega_\pm)$.

It follows from Eq. (27) in the model of a narrow ring at the point r = 0 that

$$\vartheta_0=\frac{R}{2\lambda_0\sqrt{\pi a}}\int_0^\tau \frac{Q(\tau-\tau_0)}{\tau_0^{3/2}}\exp(-\Omega)\,d\tau_0. \tag{31}$$

If the heat flux is changed at the initial instant by some amount q_0, and then remains constant, the following equations replace Eqs. (30) and (31)

$$\vartheta_0=2\frac{q_0}{\lambda_0}\left\{\sqrt{\frac{\tau a}{\pi}}\left[\exp(-\Omega_-^*)-\exp(-\Omega_+^*)\right]+\frac{R_+}{2}\ \operatorname{erfc}\sqrt{\Omega_+^*}-\frac{R_-}{2}\ \operatorname{erfc}\sqrt{\Omega_-^*}\right\}, \tag{32}$$

$$\vartheta_0=\frac{Q_0}{\lambda_0}\operatorname{erfc}\sqrt{\Omega^*}, \tag{33}$$

where

$$\Omega_\pm^*=\frac{R_\pm^2}{4a\tau};\quad \Omega^*=\frac{R^2}{4a\tau};\quad Q_0=2q_0h;\quad \operatorname{erfc}\zeta=1-\operatorname{erf}\zeta.$$

Introducing the dimensionless variables $\Theta = \vartheta_0\lambda_0/Rq_0$, $Fo = a\tau/R^2$, $\varepsilon = 2h/R$, $d_\pm = 1 \pm \varepsilon/2$, Eqs. (32) and (33) are written in the form

$$\Theta=2\sqrt{\frac{Fo}{\pi}}\left[\exp\left(-\frac{d_-^2}{4Fo}\right)-\exp\left(-\frac{d_+^2}{4Fo}\right)\right]+d_+\operatorname{erfc}\frac{d_+}{2\sqrt{Fo}}-d_-\operatorname{erfc}\frac{d_-}{2\sqrt{Fo}}, \tag{34}$$

$$\Theta=\varepsilon\operatorname{erfc}\frac{1}{2\sqrt{Fo}}. \tag{35}$$

If the Kirchhoff variable is known, the temperature may be found from it by the method proposed in [19]. This method allows the temperature to be expressed in terms of the Kirchhoff variable for any law of variation in the thermal conductivity as a function of the temperature. For some steels, however, the thermal conductivity varies linearly with temperature [20, 8], and hence the temperature at the coordinate origin is expressed in terms of the Kirchhoff variable Θ in the form

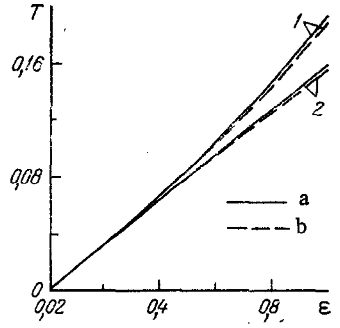

Fig. 1. Temperature variation as a function of ε according to the accurate (curve 1) and approximate (curve 2) model: a) thermosensitive body; b) nonthermosensitive body.

$$T=\frac{t_0}{t_p}=\frac{1}{k}(1-\sqrt{1-2k\Theta}), \tag{36}$$

where k = 0.132, $t_p = q_0R/\lambda_0$ is the reference temperature, $t_0 = t|_{z=0,\, r=0}$.

Calculations have been undertaken using Eq. (36), with accurate – Eq. (34) – and approximate – Eq. (35) – expressions for Θ when Fo = 0.25; the results are shown in Fig. 1, from which it follows that, when $\varepsilon \leqslant 0.2$, the results corresponding to the accurate and approximate models coincide. Taking account of the temperature dependence of the TPC in this case has little influence on the temperature distribution.

For a nonthermosensitive halfspace, replacing ϑ, λ_0 in Eqs. (32) and (33) by t, λ gives solutions corresponding to arbitrary and narrow heating rings

$$t_0=2R\frac{q_0}{\lambda}\left\{\sqrt{\frac{\mathrm{Fo}}{\pi}}\left[\exp\left(-\frac{\left(1-\frac{\varepsilon}{2}\right)^2}{4\mathrm{Fo}}\right)-\exp\left(-\frac{\left(1+\frac{\varepsilon}{2}\right)^2}{4\mathrm{Fo}}\right)\right]+\right.$$
$$\left.+\frac{1+\frac{\varepsilon}{2}}{2}\operatorname{erfc}\frac{1+\frac{\varepsilon}{2}}{2\sqrt{\mathrm{Fo}}}-\frac{1-\frac{\varepsilon}{2}}{2}\operatorname{erfc}\frac{1-\frac{\varepsilon}{2}}{2\sqrt{\mathrm{Fo}}}\right\}=2R\frac{q_0}{\lambda}\Psi(\mathrm{Fo}), \tag{37}$$

$$t_0=\frac{Q_0}{\lambda}\operatorname{erfc}\frac{1}{2\sqrt{\mathrm{Fo}}}. \tag{38}$$

With steady heat conditions (Fo → ∞), both solutions take the form

$$t_0^s=\frac{Q_0}{\lambda}. \tag{39}$$

Considering the practical aspect of the use of the solutions obtained in Eqs. (37) and (38), it may be noted that these solutions are simple expressions, in which an explicit dependence of the temperature on a whole set of TPC of the given semiinfinite body is seen. This permits nondestructive monitoring of the TPC of materials over a broad temperature range in which the TPC do not change as a function of the temperature. To determine the whole complex, there is no need to introduce temperature sensors in the internal volume of the given sample and, in addition, the temperature measurements are recorded outside the region of action of the heater. Suppose that two temperature measurements t_1 and t_2 are recorded at the coordinate origin at times τ_1 and $2\tau_1$. Then according to Eqs. (37) and (38)

$$\frac{t_1}{t_2}=\frac{\Psi(\mathrm{Fo}_1)}{\Psi(2\mathrm{Fo}_1)},\quad \frac{t_1}{t_2}=\frac{\operatorname{erfc}\frac{1}{2\sqrt{\mathrm{Fo}_1}}}{\operatorname{erfc}\frac{1}{2\sqrt{2\mathrm{Fo}_1}}}, \tag{40}$$

where $\mathrm{Fo}_1 = a\tau_1/R^2$.

It is simple to plot a graph from Eq. (40) and find Fo_1. Knowing Fo_1 and τ_1, the thermal diffusivity is determined with a specified R

$$a = \frac{Fo_1}{\tau_1} R^2. \tag{41}$$

Since Fo_1 has been found, the thermal conductivity at specified values of q_0 and R is determined

$$\lambda = 2\frac{q_0}{t_1} R\Psi(Fo_1), \quad \lambda = \frac{Q_0}{t_1} \operatorname{erfc} \frac{1}{2\sqrt{Fo_1}}.$$

If a and λ have been determined, the volume specific heat is found from the well-known formula

$$c_v = \frac{\lambda}{a}. \tag{42}$$

The TPC may also be determined as follows. Recording the temperature in steady thermal conditions, the thermal conductivity is determined from Eq. (39)

$$\lambda = \frac{Q}{t_0^s}. \tag{43}$$

Plotting graphs of the temperature variation as a function of Fo from Eq. (37) or (38) with λ in the form in Eq. (43), and measuring the temperature t_1 at time τ_1, the value of Fo_1 corresponding to t_1 is found from the graph. Since Fo_1 and τ_1 are known, the thermal diffusivity a is determined from Eq. (41). Knowing a and λ, the volume specific heat is found from Eq. (42).

For thermosensitive materials with temperature-independent thermal diffusivity, the following method may be proposed for determining the TPC. Since a is independent of the temperature over the whole range of temperature variation (including high and low temperatures), it may be determined from Eq. (41), corresponding to the temperature range in which all the TPC remain constant. Differentiating the Kirchhoff variable in Eq. (5) and its form in Eq. (35) with respect to the time, comparing the results when z = 0, r = 0, and taking into account that

$$\lambda(t_0) = ac_v(t_0) \tag{44}$$

the following expression for the volume specific heat is obtained

$$c_v(t_0) = \frac{Q_0}{\dot{t}_0 R^2} \frac{\exp\left(-\frac{1}{4Fo}\right)}{2\sqrt{\pi} Fo^{3/2}}. \tag{45}$$

In this expression, the thermal diffusivity a takes the form in Eq. (41), and the heating rate $\dot{t}_0$ is determined as follows. After recording the temperature measurements at the coordinate origin throughout the whole heating process of the sample, a graph of the variation in temperature t_0 over time is plotted. Determining $c_v(t_0)$, the thermal conductivity is obtained from Eq. (44).

Consider also the case when a time-variable heat flux of intensity

$$q(\tau) = \frac{B}{\sqrt{\tau}} S_+(\tau) \tag{46}$$

is transferred through a narrow ring; here B = const is a constant of the probe.

Substituting Eq. (46) into Eq. (31), and using handbook data [14], it is found that

$$\vartheta_0(\tau) = \frac{B_0}{\lambda_0 \sqrt{\tau}} \exp(-\Omega^*), \tag{47}$$

where $B_0 = 2hB$ is the reduced probe constant.

For a nonthermosensitive body

$$t_0(\tau) = \frac{B_0}{\lambda\sqrt{\tau}} \exp(-\Omega^*).$$

For arbitrary multiple times $n\tau$ and $2n\tau$, it follows that

$$\frac{t_0(2n\tau)}{t_0(n\tau)} = \frac{1}{\sqrt{2}}\exp\left(\frac{1}{8\mathrm{Fo}\,n}\right).$$

Hence, the Fourier number is determined

$$\mathrm{Fo} = 1/8n \ln\left[\sqrt{2}\,\frac{t_0(2n\tau)}{t_0(n\tau)}\right],$$

and then the thermal diffusivity

$$a = \frac{R^2\mathrm{Fo}}{\tau} \tag{48}$$

and the thermal conductivity

$$\lambda = \frac{B_0}{\sqrt{\tau}\, t_0(\tau)}\exp\left(-\frac{1}{4\mathrm{Fo}}\right).$$

For thermosensitive bodies, we proceed analogously to the previous case of heating. Differentiating the Kirchhoff variable in Eq. (5) when $z = 0$, $r = 0$ and its form in Eq. (47) with respect to the time and comparing the results, the thermal conductivity is found to be

$$\lambda(t_0) = \frac{B_0}{2\sqrt{\tau^3}}\left(\frac{1}{2\mathrm{Fo}} - 1\right)\frac{\exp\left(-\frac{1}{4\mathrm{Fo}}\right)}{\dot{t}_0},$$

where the thermal diffusivity takes the form in Eq. (48), and the heating rate $\dot{t}_0$ is found analogously to the preceding case of heating. Determining the thermal conductivity in this way, the volume specific heat is found from Eq. (44).

NOTATION

$\lambda(t)$, thermal conductivity; λ_0, reference thermal conductivity; $c_V(t)$, volume specific heat; $J_\nu(\zeta)$, Bessel function of the first kind of order $\nu = 0, 1$; $I_0(\zeta)$, modified zero-order Bessel function of the first kind; $\delta(\zeta)$, Dirac delta function; ${}_2F_1(-k, -1, -k; 1; r^2/\zeta^2)$, Gauss hypergeometric function; $\vartheta_3(v, x)$, theta function; erfc ζ, probability integral; s, parameter of integral Laplace transformation; ξ, parameter of integral Hankel transformation; $Q(\tau)$, reduced heat flux density; $q(\tau)$, heat flux density.

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A METHOD FOR THE APPROXIMATE SOLUTION OF A TWO-PHASE STEFAN PROBLEM WITH REVERSE MOTION OF THE FRONT

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Determination of the trajectory of a phase transition front moving in a forward or reverse direction is reduced to the solution of an ordinary differential equation. A numerical check of the results shows the method to be highly accurate.

In the design of various apparata and structures, for example, wells in regions containing frozen rocks, the operation of which leads to a change in the aggregate state of the material in the surrounding medium, one is obliged to make numerous calculations of the motion of a phase transition boundary. Use of difference methods [1-3] for these purposes leads to the expenditure of a large amount of computer time, particularly in the case in which the process involves an infinite region. In this situation expenditures of computer time increases most perceptibly when solving problems involving a reverse front owing to the fact that the boundary of the computational domain must be moved especially far away. Reduction of an infinite domain to a finite one through a change of coordinates, for example, through use of the method indicated in [4], does not in practice decrease the volume of calculations. Moreover, as computational practice shows, difference methods cease to be suitable when the temperature of the initial phase is considerably below or above the temperature of the transition phase and the development of the process proceeds at extremely slow rates. Under these circumstances the role of approximate methods in carrying out engineering calculations is enhanced, particularly methods based on L. S. Leibenzon's integral formulation of the problem [5, 6]. This formulation, when used with suitable approximations of temperature profiles, makes it possible to obtain acceptable accuracy in determining the dynamics of the front of phase transitions and, in the first place, is interesting for practical applications. The version of the integral balance method presented in [6] is more effective, in this respect, than that given in [5] since in it terms not specified by the boundary conditions were excluded. This exclusion was effected in [6] by applying an operation of double integration; however, as shown in [7], the same result can be obtained by the use of Green's transformation. This modified version of the integral balance method, when applied to a one-phase problem, guarantees high accuracy in replacing the true temperature distribution by a quasistationary one, even for large Stefan numbers [7]. Obviously, this conclusion can also be carried over to the case of the two-phase problem since in the thermal balance integral the contributions from each of the phases are taken into account independently of one another.

In what follows, a modified integral balance method is developed for the case in which the motion of the phase front commences after a preliminary initial heating of the region and also when its forward motion changes to a reverse motion after thermal action ceases. In all these cases the trajectory of the front is described by a first order ordinary differential equation.

1. First of all, we obtain the differential equation for the case of an exterior single-phase Stefan problem with convective heat exchange at the moving boundary. After dimensionalization, this problem may be reduced to solving the heat-conduction equation in the region $1 \leqslant \xi \leqslant \eta$:

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